# CHAPTER IV

# Systems with Singular Configurations

## 4.1 Introduction

The previous two chapters have developed efficient dynamics algorithms and parallel implementations of the simulation for multiple closed-chain manipulators grasping a common object. Thus far, however, the case where one or more of the manipulators are in singular configurations has not been addressed. In comparison to the body of work related to the previous chapters, much less has been completed in this area. Despite this, singularities are an important issue in robotics. They can not only pose serious problems for control algorithms that are not designed to handle them, but also cause unreasonable and erroneous behavior in numerical simulations. The latter is caused by ill-conditioned matrices that occur at and near singularities and must be inverted in the dynamics algorithms.

The algorithm in the previous chapters was inspired by the work of Lilly and Orin [24]. While they developed a sequential algorithm with a computational complexity of $O(mN)$, an equivalent approach has been developed in this work that is much more efficient for manipulators with six degrees-of-freedom but has a complexity of $O(mN^3)$. Unfortunately, they do not fully consider the case where chains may be in singular configurations. Rodriguez, Jain, and Kreutz-Delgado have also developed multiple manipulator algorithms in [30, 31]. They go on to state that a computation with an $O(m^3)$ complexity must be added to the simulation algorithm in the presence of singularities. The purpose of this chapter is to examine only those equations that are affected when manipulators become singular, and to develop algorithms with lower complexity and more efficient computations.

In particular, the equations that introduce the $O(m^3)$ computation in [30, 31] are examined, and the improved algorithm presented in this chapter introduces an $O(s^3)$ computation where $s$ is the number of chains in singular configurations. This

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represents a significant improvement over previous results. The new algorithm can also detect when manipulators are *near* singularities, and in these situations, it is also numerically very robust. In addition to this result, efficient O($m$) procedures are also presented for the cases where only one or two chains are in singular configurations. These cases are important since it is undesirable, if not uncommon and easily avoidable, to deal with systems where many of the manipulators grasping the common object are singular simultaneously. Finally, by applying the algorithm developed for the case of two singularities to a dual-arm system, the resulting algorithm requires fewer total floating point operations than that of the method presented in [24], with the added benefit of being robust in the presence of singular manipulators.

In the next section, the dynamic equations used for simulating systems of multiple manipulators are reexamined, and particular attention is given to the computational problems introduced when manipulators are in singular configurations. In Section 4.3, an efficient algorithm is developed giving special attention to the cases where only one or two manipulators are in singular configurations. In the section following, the computational complexity for this algorithm is discussed. In addition, a specific example of special interest in many robotics applications, the dual-arm system, is examined and the especially efficient computational requirements for the affected equations are listed. Finally, Section 4.5 contains a mathematical analysis which characterizes the singular solutions, and proves that the accelerations computed by the dynamics are unique under these conditions.

## 4.2 Dynamic Equations for Systems With Singular Manipulators

If a manipulator is singular, its Jacobian is rank deficient which implies that $\mathbf{\Lambda}_k^{-1}$ is also singular and cannot be inverted as required by the algorithm presented in Chapter 2. This condition affects the computation of the reference member acceleration in Eq. (2.10) and the tip forces in Eq. (2.11). These equations are repeated here for reference:

$$\left[\mathbf{I}_r + \sum_{k=1}^{m} \mathbf{\Lambda}_k\right] \mathbf{a}_r = \left[\mathbf{f}_r^g - \bar{\mathbf{f}}_r + \sum_{k=1}^{m} \mathbf{\Lambda}_k \ddot{\mathbf{x}}_{k_{open}}\right], \tag{4.1}$$

$$\mathbf{f}_k = \mathbf{\Lambda}_k \left(\ddot{\mathbf{x}}_{k_{open}} - \mathbf{a}_r\right), \qquad \text{for } k = 1, \ldots, m. \tag{4.2}$$

This section presents an algorithm, beginning with Eqs. (2.7) and (2.8), that replaces the equations above when manipulators become singular. For clarity, this approach does not begin with the efficient procedure presented in Section 2.3 for rigid grasps. However, the same analysis presented in this and subsequent sections can be carried, as desired, out by starting with Eqs. (2.16) and (2.20) in place of Eqs. (2.7) and (2.8).

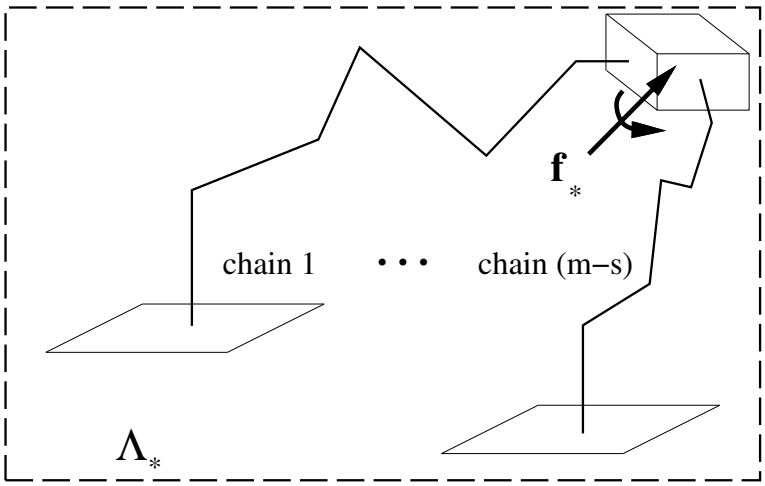


Figure 15: Dynamic variables for the nonsingular subsystem of chains.

To begin, Eqs. (2.7) and (2.8) are combined into a single block matrix equation as follows:

$$\begin{bmatrix} \mathbf{\Lambda}_1^{-1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{\Lambda}_2^{-1} & & \mathbf{0} & \mathbf{1} \\ \vdots & & \ddots & & \vdots \\ \mathbf{0} & \mathbf{0} & & \mathbf{\Lambda}_m^{-1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} & -\mathbf{I}_r \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_m \\ \mathbf{a}_r \end{bmatrix} = \begin{bmatrix} \ddot{\mathbf{x}}_{1_{open}} \\ \ddot{\mathbf{x}}_{2_{open}} \\ \vdots \\ \ddot{\mathbf{x}}_{m_{open}} \\ \bar{\mathbf{f}}_r - \mathbf{f}_r^g \end{bmatrix}, \tag{4.3}$$

where $\mathbf{1}$ is the $6 \times 6$ identity matrix and $\mathbf{0}$ is the $6 \times 6$ zero matrix. A linear system solution can then be performed on this entire matrix equation to determine the unknown tip forces and reference member acceleration. The computational complexity of this step results in the $O(m^3)$ term discussed in [30, 31]. This order of complexity is directly related to the size of the matrix in the equation to be solved which is $(6m+6) \times (6m+6)$, and which grows linearly with the number of chains in the system. This is undesirable because the cubic complexity can lead to a large amount of computation. Also, the numerical accuracy of the method may be problematic since it decreases significantly for very large systems of equations.

Before developing the efficient algorithms in this chapter which deal with systems where some of the chains are in singular configurations, some additional notation is defined. First, the chains are numbered so that chains $1, \ldots, (m-s)$ are the nonsingular manipulators. Then, the following two quantities illustrated in Figure 15

are defined:

$$\mathbf{\Lambda}_* = \mathbf{I}_r + \sum_{k=1}^{m-s} \mathbf{\Lambda}_k, \tag{4.4}$$

$$\mathbf{f}_* = \mathbf{f}_r^g - \bar{\mathbf{f}}_r + \sum_{k=1}^{m-s} \mathbf{\Lambda}_k \ddot{\mathbf{x}}_{k_{open}}. \tag{4.5}$$

The first quantity, $\mathbf{\Lambda}_*$, can be thought of as the operational space inertia matrix of the nonsingular subsystem of chains which consists of the $m - s$ nonsingular manipulators and the reference member, and is computed at the reference point described in Figure 2. Likewise, $\mathbf{f}_*$ is the sum of all of the forces of this subsystem that are exerted on the reference member at the reference point.

From these two quantities, the acceleration of the reference member without the interaction of the singular chains (called the open, nonsingular subsystem acceleration) would be determined by $\mathbf{\Lambda}_*^{-1}\mathbf{f}_*$, which is equivalent to Eq. (4.1) when $s = 0$. This implies that the computation involving the nonsingular chains will be unmodified, and therefore, introduces no additional computation in the algorithm from Chapter 2. What is modified, however, is the computation of the forces at the tips of the singular chains and the reference member acceleration. The following section describes an efficient way to perform these computations.

## 4.3 Efficient Singularity Algorithm

In this section, algorithms for the computation of the reference member acceleration and the tip forces of each chain are presented for systems with $m$ manipulators where an arbitrary number of them are in singular configurations. These algorithms have a lower complexity than the one described in the previous section which solved Eq. (4.3) directly. In the first two parts, algorithms are presented for the cases when one and two manipulators in a general $m$-manipulator system are singular. Then, these results are extended to the case where the system has an arbitrary number of singular chains, $s$.

### 4.3.1 One Singular Chain

For an $m$-manipulator system with one singular chain (chain $m$), Eq. (4.3) is rearranged as follows to isolate the singular chain:

$$\begin{bmatrix} \boldsymbol{\Lambda}_1^{-1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}_2^{-1} & & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \vdots & & \ddots & & \vdots & \\ \mathbf{0} & \mathbf{0} & & \boldsymbol{\Lambda}_{m-1}^{-1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} & -\mathbf{I}_r & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & & \mathbf{0} & \mathbf{1} & \boldsymbol{\Lambda}_m^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_{m-1} \\ \mathbf{a}_r \\ \mathbf{f}_m \end{bmatrix} = \begin{bmatrix} \ddot{\mathbf{x}}_{1_{open}} \\ \ddot{\mathbf{x}}_{2_{open}} \\ \vdots \\ \ddot{\mathbf{x}}_{m-1_{open}} \\ \bar{\mathbf{f}}_r - \mathbf{f}_r^g \\ \ddot{\mathbf{x}}_{m_{open}} \end{bmatrix}. \tag{4.6}$$

An efficient block elimination approach is then used to begin reducing this system. The inverse of chain $m$'s operational space inertia matrix is singular and, therefore, positive semidefinite. The remainder are strictly positive definite, and can be used to eliminate the first $m-1$ identity matrices in the $m$th row. This is accomplished by premultiplying row $k$ of Eq. (4.6) by $\boldsymbol{\Lambda}_k$ and subtracting it from the $m$th row (for $k = 1, \ldots, m-1$). This results in the following system of equations:

$$\begin{bmatrix} \boldsymbol{\Lambda}_1^{-1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}_2^{-1} & & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \vdots & & \ddots & & \vdots & \\ \mathbf{0} & \mathbf{0} & & \boldsymbol{\Lambda}_{m-1}^{-1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -\boldsymbol{\Lambda}_* & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & & \mathbf{0} & \mathbf{1} & \boldsymbol{\Lambda}_m^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_{m-1} \\ \mathbf{a}_r \\ \mathbf{f}_m \end{bmatrix} = \begin{bmatrix} \ddot{\mathbf{x}}_{1_{open}} \\ \ddot{\mathbf{x}}_{2_{open}} \\ \vdots \\ \ddot{\mathbf{x}}_{m-1_{open}} \\ -\mathbf{f}_* \\ \ddot{\mathbf{x}}_{m_{open}} \end{bmatrix}, \tag{4.7}$$

where $\boldsymbol{\Lambda}_*$ and $\mathbf{f}_*$ are the quantities defined in the previous section with $s = 1$. This elimination step has accumulated dynamic quantities related to the nonsingular subsystem, namely its inertia and the forces acting upon it, and decoupled its dynamic equation as well as the singular chain's from the remainder of the system so that a much smaller system of equations needs to be solved.

This system can be reduced further, since $\boldsymbol{\Lambda}_*$ is positive definite (it is the sum of positive definite matrices), and therefore, nonsingular. Using its inverse, the equations in the last two rows are reduced to the following:

$$\begin{bmatrix} -\boldsymbol{\Lambda}_* & \mathbf{1} \\ \mathbf{0} & \boldsymbol{\Lambda}_m^{-1} + \boldsymbol{\Lambda}_*^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{a}_r \\ \mathbf{f}_m \end{bmatrix} = \begin{bmatrix} -\mathbf{f}_* \\ \ddot{\mathbf{x}}_{m_{open}} - \boldsymbol{\Lambda}_*^{-1}\mathbf{f}_* \end{bmatrix}. \tag{4.8}$$

Because $\boldsymbol{\Lambda}_m^{-1}$ is positive semidefinite and $\boldsymbol{\Lambda}_*^{-1}$ is positive definite, their sum is also positive definite, and therefore, invertible. As a result, a solution to the system may be obtained by performing back-substitution on the block matrix equations defined

by Eq. (4.8) and the first $m-1$ rows of Eq. (4.7). This results in the following algorithm:

$$\mathbf{f}_m = \left(\mathbf{\Lambda}_m^{-1} + \mathbf{\Lambda}_*^{-1}\right)^{-1} \left(\ddot{\mathbf{x}}_{m_{open}} - \mathbf{\Lambda}_*^{-1}\mathbf{f}_*\right), \tag{4.9}$$

$$\mathbf{a}_r = \mathbf{\Lambda}_*^{-1}\left(\mathbf{f}_m + \mathbf{f}_*\right), \tag{4.10}$$

$$\mathbf{f}_k = \mathbf{\Lambda}_k\left(\ddot{\mathbf{x}}_{k_{open}} - \mathbf{a}_r\right) \qquad \text{for all} \quad k = 1, \ldots, m-1. \tag{4.11}$$

Note that there is always a unique solution for the motion of an object in a physical system which means that the solution for $\mathbf{a}_r$ will be unique. Since $\mathbf{\Lambda}_m^{-1} + \mathbf{\Lambda}_*^{-1}$ is also invertible, there will always be a unique solution for the tip forces in a system with one singular chain as well.

### 4.3.2 Two Singular Chains

With two singular chains ($s = 2$), the block matrix equation is again rewritten by moving the singular inverse operational space inertia matrices (for chains $m-1$ and $m$) to the bottom as follows:

$$\begin{bmatrix} \mathbf{\Lambda}_1^{-1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_2^{-1} & & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \vdots & & \ddots & & & \vdots & \\ \mathbf{0} & \mathbf{0} & & \mathbf{\Lambda}_{m-2}^{-1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & & \mathbf{1} & -\mathbf{I}_r & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{1} & \mathbf{\Lambda}_{m-1}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{\Lambda}_m^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_{m-2} \\ \mathbf{a}_r \\ \mathbf{f}_{m-1} \\ \mathbf{f}_m \end{bmatrix} = \begin{bmatrix} \ddot{\mathbf{x}}_{1_{open}} \\ \ddot{\mathbf{x}}_{2_{open}} \\ \vdots \\ \ddot{\mathbf{x}}_{m-2_{open}} \\ \bar{\mathbf{f}}_r - \mathbf{f}_r^g \\ \ddot{\mathbf{x}}_{m-1_{open}} \\ \ddot{\mathbf{x}}_{m_{open}} \end{bmatrix}, \tag{4.12}$$

As in the previous section, the first $m-2$ identity matrices in the $(m-1)$st row are eliminated by taking the operational space inertia matrices of the first $m-2$ chains and performing block eliminations. As a result, the last three equations can be decoupled from the others and written as follows:

$$\begin{bmatrix} -\mathbf{\Lambda}_* & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{\Lambda}_{m-1}^{-1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{\Lambda}_m^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{a}_r \\ \mathbf{f}_{m-1} \\ \mathbf{f}_m \end{bmatrix} = \begin{bmatrix} -\mathbf{f}_* \\ \ddot{\mathbf{x}}_{m-1_{open}} \\ \ddot{\mathbf{x}}_{m_{open}} \end{bmatrix}. \tag{4.13}$$

Because $\mathbf{\Lambda}_*$ is the sum of all of the positive definite inertia matrices, it will always be positive definite, and can be used to eliminate the two identity matrices in the first column. The resulting system of equations appears as follows:

$$\begin{bmatrix} -\mathbf{\Lambda}_* & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{\Lambda}_{m-1}^{-1} + \mathbf{\Lambda}_*^{-1} & \mathbf{\Lambda}_*^{-1} \\ \mathbf{0} & \mathbf{\Lambda}_*^{-1} & \mathbf{\Lambda}_m^{-1} + \mathbf{\Lambda}_*^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{a}_r \\ \mathbf{f}_{m-1} \\ \mathbf{f}_m \end{bmatrix} = \begin{bmatrix} -\mathbf{f}_* \\ \ddot{\mathbf{x}}_{m-1_{open}} - \mathbf{\Lambda}_*^{-1}\mathbf{f}_* \\ \ddot{\mathbf{x}}_{m_{open}} - \mathbf{\Lambda}_*^{-1}\mathbf{f}_* \end{bmatrix}. \tag{4.14}$$

The matrix, $\mathbf{\Lambda}_{m-1}^{-1} + \mathbf{\Lambda}_*^{-1}$ (or $\mathbf{\Lambda}_m^{-1} + \mathbf{\Lambda}_*^{-1}$ for that matter), is nonsingular, and can be used to reduce the last $2 \times 2$ block of matrices. The resulting equation appears as follows:

$$\begin{bmatrix} -\mathbf{\Lambda}_* & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{\Lambda}_{m-1}^{-1} + \mathbf{\Lambda}_*^{-1} & \mathbf{\Lambda}_*^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Lambda}_m^{-1} + \mathbf{\Lambda}_*^{-1} - \mathbf{\Lambda}_*^{-1}\left(\mathbf{\Lambda}_{m-1}^{-1} + \mathbf{\Lambda}_*^{-1}\right)^{-1}\mathbf{\Lambda}_*^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{a}_r \\ \mathbf{f}_{m-1} \\ \mathbf{f}_m \end{bmatrix} = \begin{bmatrix} -\mathbf{f}_* \\ \ddot{\mathbf{x}}_{m-1_{open}} - \mathbf{\Lambda}_*^{-1}\mathbf{f}_* \\ \ddot{\mathbf{x}}_{m_{open}} - \mathbf{\Lambda}_*^{-1}\mathbf{f}_* - \mathbf{\Lambda}_*^{-1}\left(\mathbf{\Lambda}_{m-1}^{-1} + \mathbf{\Lambda}_*^{-1}\right)^{-1}\left(\ddot{\mathbf{x}}_{m-1_{open}} - \mathbf{\Lambda}_*^{-1}\mathbf{f}_*\right) \end{bmatrix}. \quad (4.15)$$

The rank deficiency (dimension of the null space) in the original block matrix equation of Eq. (4.3) will be equal to the rank deficiency in the lower right block of Eq. (4.15). If this block is nonsingular, then a unique solution can be found for $\mathbf{a}_r$ and all of the $\mathbf{f}_k$'s using the following equations:

$$\begin{aligned} \mathbf{f}_m &= \left[\mathbf{\Lambda}_m^{-1} + \mathbf{\Lambda}_*^{-1} - \mathbf{\Lambda}_*^{-1}\left(\mathbf{\Lambda}_{m-1}^{-1} + \mathbf{\Lambda}_*^{-1}\right)^{-1}\mathbf{\Lambda}_*^{-1}\right]^{-1}\left[\ddot{\mathbf{x}}_{m_{open}} - \mathbf{\Lambda}_*^{-1}\mathbf{f}_* - \right. \\ &\qquad \left. \mathbf{\Lambda}_*^{-1}\left(\mathbf{\Lambda}_{m-1}^{-1} + \mathbf{\Lambda}_*^{-1}\right)^{-1}\left(\ddot{\mathbf{x}}_{m-1_{open}} - \mathbf{\Lambda}_*^{-1}\mathbf{f}_*\right)\right], \quad (4.16) \\ \mathbf{f}_{m-1} &= \left(\mathbf{\Lambda}_{m-1}^{-1} + \mathbf{\Lambda}_*^{-1}\right)^{-1}\left(\ddot{\mathbf{x}}_{m-1_{open}} - \mathbf{\Lambda}_*^{-1}\mathbf{f}_* - \mathbf{\Lambda}_*^{-1}\mathbf{f}_m\right), \quad (4.17) \\ \mathbf{a}_r &= \mathbf{\Lambda}_*^{-1}\left(\mathbf{f}_{m-1} + \mathbf{f}_m + \mathbf{f}_*\right), \quad (4.18) \\ \mathbf{f}_k &= \mathbf{\Lambda}_k\left(\ddot{\mathbf{x}}_{k_{open}} - \mathbf{a}_r\right) \qquad \text{for all} \quad k = 1, \ldots, m-2. \quad (4.19) \end{aligned}$$

If, on the other hand, the lower right block in the matrix of Eq. (4.15) is singular, the rigid body dynamics of the system of manipulators implies that the matrix equation will still be consistent so that solutions exist. In this case, however, an infinite number of solutions will exist for the tip forces. At the same time, only one solution will be found for the motion ($\mathbf{a}_r$) in the system. A proof of these statements is given in Section 4.5.

This can also be intuitively understood because one or more components of the force vector, $\mathbf{f}_m$, will be opposed by an infinite inertia that is created by a combination of the nonsingular subsystem and the singular chain $m-1$. These components will also see an infinite inertia from chain $m$ as well, and regardless of the value assigned to them, no motion will result. In this case, the singularities of both chains can be thought of as having "lined-up." Mathematically, the null spaces of $\mathbf{J}_{m-1}^T$ and $\mathbf{J}_m^T$ overlap as shown in Section 4.5. Note, however, that values given to these components of $\mathbf{f}_m$ will be negated by corresponding components in $\mathbf{f}_{m-1}$.

To solve for $\mathbf{f}_m$, in this case, a linear system solution with full pivoting can be employed which will produce one or more zero rows that correspond to the

components of this force that can be set to any value. For convenience, they are set to zero without affecting the resulting acceleration of the reference member or any of the chains' joints. The remaining components of $\mathbf{f}_m$ can then be determined, and the solution for the remaining forces and reference member acceleration proceeds as specified in Eqs. (4.17), (4.18), and (4.19).

### 4.3.3 More Than Two Singular Chains

By extending this procedure to a system of $m$ chains with an arbitrary number of singular chains ($3 \leq s \leq m$), the following set of equations need to be solved to find the required tip forces exerted by the singular chains onto the reference member:

$$\begin{bmatrix} \boldsymbol{\Lambda}_{m-s+1}^{-1} + \boldsymbol{\Lambda}_*^{-1} & \boldsymbol{\Lambda}_*^{-1} & \cdots & \boldsymbol{\Lambda}_*^{-1} \\ \boldsymbol{\Lambda}_*^{-1} & \boldsymbol{\Lambda}_{m-s+2}^{-1} + \boldsymbol{\Lambda}_*^{-1} & & \boldsymbol{\Lambda}_*^{-1} \\ \vdots & & \ddots & \vdots \\ \boldsymbol{\Lambda}_*^{-1} & \boldsymbol{\Lambda}_*^{-1} & \cdots & \boldsymbol{\Lambda}_m^{-1} + \boldsymbol{\Lambda}_*^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{f}_{m-s+1} \\ \mathbf{f}_{m-s+2} \\ \vdots \\ \mathbf{f}_m \end{bmatrix}$$

$$= \begin{bmatrix} \ddot{\mathbf{x}}_{m-s+1_{open}} - \boldsymbol{\Lambda}_*^{-1}\mathbf{f}_* \\ \ddot{\mathbf{x}}_{m-s+2_{open}} - \boldsymbol{\Lambda}_*^{-1}\mathbf{f}_* \\ \vdots \\ \ddot{\mathbf{x}}_{m_{open}} - \boldsymbol{\Lambda}_*^{-1}\mathbf{f}_* \end{bmatrix}. \quad (4.20)$$

Then, the following set of equations are used to compute the remainder of the unknowns:

$$\mathbf{a}_r = \boldsymbol{\Lambda}_*^{-1} \left( \sum_{k=1}^{s} \mathbf{f}_{m-k+1} + \mathbf{f}_* \right), \quad (4.21)$$

$$\mathbf{f}_k = \boldsymbol{\Lambda}_k \left( \ddot{\mathbf{x}}_{k_{open}} - \mathbf{a}_r \right) \qquad \text{for all} \quad k = 1, \ldots, m-s. \quad (4.22)$$

Because $\boldsymbol{\Lambda}_{m-s+1}^{-1} + \boldsymbol{\Lambda}_*^{-1}$ is positive definite (using the same argument as before), it can be inverted and used to reduce the size of the block matrix equation that needs to be solved. The resulting size of the block matrix in this equation is $6(s-1) \times 6(s-1)$. This equation is then used to compute the values for $\mathbf{f}_{m-s+2}, \ldots, \mathbf{f}_m$, and the following equation is used to compute $\mathbf{f}_{m-s+1}$:

$$\mathbf{f}_{m-s+1} = \left( \boldsymbol{\Lambda}_{m-s+1}^{-1} + \boldsymbol{\Lambda}_*^{-1} \right)^{-1} \left( \ddot{\mathbf{x}}_{m-s+1_{open}} - \boldsymbol{\Lambda}_*^{-1}\mathbf{f}_* - \boldsymbol{\Lambda}_*^{-1} \sum_{k=1}^{s-1} \mathbf{f}_{m-k+1} \right). \quad (4.23)$$

By the same arguments that were given in the previous section, the smaller block matrix system of equations for finding $\mathbf{f}_{m-s+2}, \ldots, \mathbf{f}_m$, will always have at least one solution, but could have an infinite number of them. Therefore, a method for the solution of this linear system should again employ full pivoting to identify the free components of the force vectors (if any), and set them to zero.

## 4.4 Computational Requirements

Chapter 2 presented an efficient algorithm for the evaluation of $\mathbf{\Lambda}_k^{-1}$ and $\ddot{\mathbf{x}}_{k_{open}}$ for each chain in the system. When none of these chains are in singular configurations, the original algorithm may be used which consists of Eqs. (4.1) and (4.2). They require $m$ inverses (of size $6 \times 6$) to find the operational space inertia matrices for all of the manipulators, followed by a $6 \times 6$ linear system solution to find $\mathbf{a}_r$. The computation required to compute these quantities ($\mathbf{a}_r$ and the $\mathbf{f}_k$'s) has a complexity of $\mathrm{O}(m)$ which does not change the overall complexity of the dynamic simulation algorithm.

With the new algorithms presented in this chapter and the same system of manipulators, the inverses for all $m$ operational space inertia matrices must still be attempted in order to identify the singular or nearly singular chains (the ones with inordinately large condition numbers). The number of such matrices indicates which of the procedures from the previous section should be used to compute the tip forces and acceleration of the reference member.

All of these methods require an additional $6 \times 6$ inverse of the positive definite $\mathbf{\Lambda}_*$ matrix at a cost of [141M, 90A] [37]. For the case with one singular chain, a linear system solution is used to compute the force exerted at the tip of the singular chain, and a matrix-vector multiply is used to compute the acceleration of the reference member. Note that this is the opposite of what is done in the original algorithm. Computation of the rest of the forces proceeds as before.

With two singular chains, the inverse of $\mathbf{\Lambda}_{m-1}^{-1} + \mathbf{\Lambda}_*^{-1}$ can be performed. However, less computation is required if a root-free Cholesky decomposition is performed. This decomposition requires $[(\frac{1}{6}n^3 + \frac{1}{2}n^2 - \frac{2}{3}n)\mathrm{M}, (\frac{1}{6}n^3 - \frac{1}{6}n)\mathrm{A}]$ for an $n \times n$ matrix which is [50M, 35A] when $n = 6$. To perform a matrix-vector multiply with the inverse, one back-substitution is used requiring $[(n^2)\mathrm{M}, (n^2 - n)\mathrm{A}]$, or [36M, 30A] for $n = 6$ [37]. Matrix-matrix multiplies are accomplished with six back substitutions (one for each column of the matrix to be multiplied). Therefore linear systems solutions are required for both, $\mathbf{f}_m$ and $\mathbf{f}_{m-1}$, and a matrix-vector multiply is used to compute the acceleration.

In the unlikely case when more than two chains are singular, the last part of the previous section shows that the size of the linear system to be solved, $(6s - 6) \times (6s - 6)$, grows linearly with $s$. This solution requires an elimination method with full pivoting which has computational complexity of $\mathrm{O}(s^3)$. This is a significant improvement over the previous $\mathrm{O}(m^3)$ results where the size of the matrix is $(6m + 6) \times (6m + 6)$. As before, a $6 \times 6$ linear system solution yields $\mathbf{f}_{m-s+1}$, and a matrix-vector multiply computes the acceleration.

Table 7: Methods for computation of the dual-arm system.

Given: $\mathbf{I}_r$ and $\mathbf{I}_r^{-1}$ (constant, diagonal), $\bar{\mathbf{f}}_r - \mathbf{f}_r^g$, and

$\mathbf{\Lambda}_k^{-1}$, $\ddot{\mathbf{x}}_{k_{open}}$, for $k = 1, 2$.

Method I: (from Chapter 2)

$$\begin{aligned} \mathbf{a}_r &= (\mathbf{\Lambda}_1 + \mathbf{\Lambda}_2 + \mathbf{I}_r)^{-1} \left[ \mathbf{\Lambda}_1 \ddot{\mathbf{x}}_{1_{open}} + \mathbf{\Lambda}_2 \ddot{\mathbf{x}}_{2_{open}} - \left( \bar{\mathbf{f}}_r - \mathbf{f}_r^g \right) \right], \\ \mathbf{f}_k &= \mathbf{\Lambda}_k \left( \ddot{\mathbf{x}}_{k_{open}} - \mathbf{a}_r \right) \qquad k = 1, 2. \end{aligned}$$

Method II: (this chapter)

$$\begin{aligned} \mathbf{f}_2 &= \left[ \mathbf{\Lambda}_2^{-1} + \mathbf{I}_r^{-1} - \mathbf{I}_r^{-1} \left( \mathbf{\Lambda}_1^{-1} + \mathbf{I}_r^{-1} \right)^{-1} \mathbf{I}_r^{-1} \right]^{-1} \cdot \\ &\quad \left\{ \ddot{\mathbf{x}}_{2_{open}} + \mathbf{I}_r^{-1} \left( \bar{\mathbf{f}}_r - \mathbf{f}_r^g \right) - \mathbf{I}_r^{-1} \left( \mathbf{\Lambda}_1^{-1} + \mathbf{I}_r^{-1} \right)^{-1} \left[ \ddot{\mathbf{x}}_{1_{open}} + \mathbf{I}_r^{-1} \left( \bar{\mathbf{f}}_r - \mathbf{f}_r^g \right) \right] \right\} \\ \mathbf{f}_1 &= \left( \mathbf{\Lambda}_1^{-1} + \mathbf{I}_r^{-1} \right)^{-1} \left[ \ddot{\mathbf{x}}_{1_{open}} + \mathbf{I}_r^{-1} \left( \bar{\mathbf{f}}_r - \mathbf{f}_r^g \right) - \mathbf{I}_r^{-1} \mathbf{f}_2 \right] \\ \mathbf{a}_r &= \mathbf{I}_r^{-1} \mathbf{f}_1 + \mathbf{I}_r^{-1} \mathbf{f}_2 - \mathbf{I}_r^{-1} \left( \bar{\mathbf{f}}_r - \mathbf{f}_r^g \right) \end{aligned}$$

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### 4.4.1 Special Case: The Dual-Arm System

Another interesting and useful result can be obtained from the solution procedure presented in this chapter when a dual-arm system is simulated. This case is of particular interest because many applications for such a system have been cited and a significant amount of research has been performed in this area [42]. A comparison of the results using the method summarized in Section 2 [Eqs. (4.1) and (4.2)] and that presented in the section discussing two singular chains [Eqs. (4.16), (4.17), and (4.18)] is discussed in this section. The two sets of equations are referred to as Method I and Method II, respectively.

In Table 7, the equations for these methods have been rewritten specifically for the dual-arm system. It has been assumed that efficient algorithms have been employed to compute the open-chain accelerations and the inverse operational space inertia matrices for both manipulators, and only the computations to find the forces at the tips and the acceleration of the reference member are examined here. When Method II is employed for this system, no checking for chains in singular configura-

Table 8: Computational requirements for Method 1.

| Computation | | × | + |
|---|---|---|---|
| $\left(\mathbf{\Lambda}_k^{-1}\right)^{-1}$ | $k = 1, 2$ | 282 | 180 |
| $\mathbf{\Lambda}_k \ddot{\mathbf{x}}_{k_{open}}$ | $k = 1, 2$ | 72 | 60 |
| $\sum_{k=1}^{2} \mathbf{\Lambda}_k \ddot{\mathbf{x}}_{k_{open}} - \left(\bar{\mathbf{f}}_r - \mathbf{f}_r^g\right)$ | | | 12 |
| $\mathbf{\Lambda}_1 + \mathbf{\Lambda}_2 + \mathbf{I}_r$ | | | 27 |
| $\mathbf{a}_r$ | | 86 | 65 |
| $\ddot{\mathbf{x}}_{k_{open}} - \mathbf{a}_r$ | $k = 1, 2$ | | 12 |
| $\mathbf{f}_k$ | $k = 1, 2$ | 72 | 60 |
| Total: | | 512 | 416 |

tions needs to be done before these equations are employed. Instead, the operational space inertia matrices of the two manipulators are treated as if they were singular. As a consequence, the terms, $\mathbf{\Lambda}_*$ and $\mathbf{f}_*$, simplify to the reference member inertia, $\mathbf{I}_r$, and a force term, $\mathbf{f}_r^g - \bar{\mathbf{f}}_r$.

Although the equations for Method I appear to be much simpler, detailed inspection of the methods indicates that Method II can be performed using significantly fewer computations when the reference point is placed at the center of gravity of the reference member, and the axes of the coordinate frame correspond to the reference member's principal axes. Under these circumstances, the inertia matrix for the reference member, $\mathbf{I}_r$, will be diagonal as will its inverse. Since this coordinate frame is also fixed to the reference member, this quantity is constant and can be computed off-line.

The computation required for both methods can then be reduced in certain steps by using the knowledge that the inertia matrix, $\mathbf{I}_r$ is diagonal, and the operational space inertia matrices are symmetric. One significant observation is that it becomes much less costly to compute the inverse of $\mathbf{\Lambda}_1^{-1} + \mathbf{I}_r^{-1}$ directly, rather than performing the Cholesky decomposition and back-substitution, because the latter involves the diagonal matrix, $\mathbf{I}_r^{-1}$, and it is more efficient to perform matrix-matrix multiplies instead. This results in the total number of operations shown in Tables 8 and 9. As a result Method II reduces the amount of computation by approximately 30%. In addition to this computational advantage, Method II is more robust because it will also work correctly and accurately in situations when one or both arms are at or

Table 9: Computational requirements for Method 2.

| Computation | | × | + |
|---|---|---|---|
| $\mathbf{\Lambda}_k^{-1} + \mathbf{I}_r^{-1}$ | $k = 1, 2$ | | 12 |
| $\mathbf{I}_r^{-1}\left(\bar{\mathbf{f}}_r - \mathbf{f}_r^g\right)$ | | 6 | |
| $\ddot{\mathbf{x}}_{k_{open}} + \mathbf{I}_r^{-1}\left(\bar{\mathbf{f}}_r - \mathbf{f}_r^g\right)$ | $k = 1, 2$ | | 12 |
| $\left(\mathbf{\Lambda}_1^{-1} + \mathbf{I}_r^{-1}\right)^{-1}$ ($6 \times 6$ inverse) | | 141 | 90 |
| $\mathbf{I}_r^{-1}\left(\mathbf{\Lambda}_1^{-1} + \mathbf{I}_r^{-1}\right)^{-1}$ | | 36 | |
| $\mathbf{I}_r^{-1}\left(\mathbf{\Lambda}_1^{-1} + \mathbf{I}_r^{-1}\right)^{-1}\mathbf{I}_r^{-1}$ | | 21 | |
| $\mathbf{I}_r^{-1}\left(\mathbf{\Lambda}_1^{-1} + \mathbf{I}_r^{-1}\right)^{-1}\left[\ddot{\mathbf{x}}_{1_{open}} + \mathbf{I}_r^{-1}\left(\bar{\mathbf{f}}_r - \mathbf{f}_r^g\right)\right]$ | | 36 | 30 |
| $\left[\ddot{\mathbf{x}}_{2_{open}} + \mathbf{I}_r^{-1}\left(\bar{\mathbf{f}}_r - \mathbf{f}_r^g\right)\right] - \mathbf{I}_r^{-1}\left(\mathbf{\Lambda}_1^{-1} + \mathbf{I}_r^{-1}\right)^{-1}\left[\ddot{\mathbf{x}}_{1_{open}} + \mathbf{I}_r^{-1}\left(\bar{\mathbf{f}}_r - \mathbf{f}_r^g\right)\right]$ | | | 6 |
| $\left(\mathbf{\Lambda}_2^{-1} + \mathbf{I}_r^{-1}\right) - \mathbf{I}_r^{-1}\left(\mathbf{\Lambda}_1^{-1} + \mathbf{I}_r^{-1}\right)^{-1}\mathbf{I}_r^{-1}$ | | | 21 |
| $\mathbf{f}_2$ ($6 \times 6$ linear system solution) | | 86 | 65 |
| $\mathbf{I}_r^{-1}\mathbf{f}_2$ | | 6 | |
| $\left[\ddot{\mathbf{x}}_{1_{open}} + \mathbf{I}_r^{-1}\left(\bar{\mathbf{f}}_r - \mathbf{f}_r^g\right)\right] - \mathbf{I}_r^{-1}\mathbf{f}_2$ | | | 6 |
| $\mathbf{f}_1$ | | 36 | 30 |
| $\mathbf{I}_r^{-1}\mathbf{f}_1$ | | 6 | |
| $\mathbf{a}_r$ | | | 12 |
| Total: | | 374 | 284 |

near singularities.

## 4.5 Singularity Conditions

In this section, the case when the dynamic equations of motion result in a singular system of equations is examined. The conditions under which the system matrix becomes singular and its null space are characterized in the first part. Then, in the second part, it is shown that the reference member acceleration will be unique even when the manipulator tip forces are not.

### 4.5.1 Characterization of Singular Solutions

In this section, solutions to the dynamic equations are characterized for the case when two manipulators are in singular configurations. This problem reduces to showing that the linear system of equations, $\mathbf{Ax} = \mathbf{b}$, has a solution when $\mathbf{A}$ and $\mathbf{b}$ are taken from the last line of Eq. (4.15):

$$\mathbf{A} = \mathbf{\Lambda}_m^{-1} + \mathbf{\Lambda}_*^{-1} - \mathbf{\Lambda}_*^{-1}\left(\mathbf{\Lambda}_*^{-1} + \mathbf{\Lambda}_{m-1}^{-1}\right)^{-1}\mathbf{\Lambda}_*^{-1}, \text{ and} \tag{4.24}$$

$$\mathbf{b} = \ddot{\mathbf{x}}_{m_{open}} - \mathbf{\Lambda}_*^{-1}\mathbf{f}_* - \mathbf{\Lambda}_*^{-1}\left(\mathbf{\Lambda}_*^{-1} + \mathbf{\Lambda}_{m-1}^{-1}\right)^{-1}\left(\ddot{\mathbf{x}}_{m-1_{open}} - \mathbf{\Lambda}_*^{-1}\mathbf{f}_*\right). \tag{4.25}$$

In these equations, $\mathbf{\Lambda}_{m-1}^{-1}$ and $\mathbf{\Lambda}_m^{-1}$, are symmetric, positive semidefinite matrices, and $\mathbf{\Lambda}_*^{-1}$ is symmetric, positive definite. When $\mathbf{A}$ is nonsingular there will always be a unique solution. The next step is to determine the conditions under which $\mathbf{A}$ becomes singular.

Note from Eqs. (2.5) and (2.6) that $\mathbf{\Lambda}_{m-1}^{-1} = \mathbf{J}_{m-1}\mathbf{H}_{m-1}^{-1}\mathbf{J}_{m-1}^T$, where $\mathbf{H}_{m-1}^{-1}$ is also symmetric and positive definite, but $\mathbf{J}_{m-1}$ is rank deficient. Therefore, Eq. (4.24) can be rewritten as follows:

$$\mathbf{A} = \mathbf{\Lambda}_m^{-1} + \mathbf{\Lambda}_*^{-1} - \mathbf{\Lambda}_*^{-1}\left(\mathbf{\Lambda}_*^{-1} + \mathbf{J}_{m-1}\mathbf{H}_{m-1}^{-1}\mathbf{J}_{m-1}^T\right)^{-1}\mathbf{\Lambda}_*^{-1}. \tag{4.26}$$

The term inside the parentheses can now be rewritten by applying the following lemma:

**Lemma 1 (Woodbury)** [40] *If* $\mathbf{P}$, $\mathbf{Q}$, *and* $\mathbf{Q}^{-1} + \mathbf{RP}^{-1}\mathbf{S}$ *are nonsingular then the following holds:*

$$(\mathbf{P} + \mathbf{SQR})^{-1} = \mathbf{P}^{-1} - \mathbf{P}^{-1}\mathbf{S}\left(\mathbf{Q}^{-1} + \mathbf{RP}^{-1}\mathbf{S}\right)^{-1}\mathbf{RP}^{-1} \tag{4.27}$$

*The proof can be completed by multiplying the right hand side by* $(\mathbf{P} + \mathbf{SQR})$ *and, with some manipulation, showing that it is equal to the identity matrix.*

The $\mathbf{A}$ matrix can then be written in the following form:

$$\mathbf{A} = \mathbf{\Lambda}_m^{-1} + \mathbf{\Lambda}_*^{-1} - \mathbf{\Lambda}_*^{-1}\left[\mathbf{\Lambda}_* - \mathbf{\Lambda}_*\mathbf{J}_{m-1}\left(\mathbf{H}_{m-1} + \mathbf{J}_{m-1}^T\mathbf{\Lambda}_*\mathbf{J}_{m-1}\right)^{-1}\mathbf{J}_{m-1}^T\mathbf{\Lambda}_*\right]\mathbf{\Lambda}_*^{-1}. \tag{4.28}$$

This simplifies to:

$$\mathbf{A} = \mathbf{\Lambda}_m^{-1} + \mathbf{J}_{m-1}\left(\mathbf{H}_{m-1} + \mathbf{J}_{m-1}^T\mathbf{\Lambda}_*\mathbf{J}_{m-1}\right)^{-1}\mathbf{J}_{m-1}^T, \tag{4.29}$$

$$= \mathbf{J}_m\mathbf{H}_m^{-1}\mathbf{J}_m^T + \mathbf{J}_{m-1}\left(\mathbf{H}_{m-1} + \mathbf{J}_{m-1}^T\mathbf{\Lambda}_*\mathbf{J}_{m-1}\right)^{-1}\mathbf{J}_{m-1}^T. \tag{4.30}$$

Since $\mathbf{H}_m^{-1}$ and the term inside the parentheses are symmetric and positive definite, $\mathbf{A}$ is also symmetric, and *usually* positive definite. The only time that $\mathbf{A}$ is singular is when there exists a vector which lies in the null spaces of both $\mathbf{J}_{m-1}^T$ and $\mathbf{J}_m^T$. When this is the case, then it can be formally stated that $\mathbf{A}^T\mathbf{v} = \mathbf{A}\mathbf{v} = \mathbf{0} \;\; \forall \; \mathbf{v} \in N(\mathbf{J}_{m-1}^T) \cap N(\mathbf{J}_m^T)$.

To also show the existence of solutions even when $\mathbf{A}$ is singular, a set of vectors, $\mathbf{v}$, must satisfy the conditions in the following theorem which is stated without proof:

**Theorem 1 (Fredholm Alternative)** [43] *The equation,* $\mathbf{A}\mathbf{x} = \mathbf{b}$ *has a solution if and only if* $\mathbf{v}^T\mathbf{b} = \mathbf{0}$ $\forall \mathbf{v}$ *that satisfies* $\mathbf{A}^T\mathbf{v} = \mathbf{0}$.

Therefore, $\mathbf{v}^T\mathbf{b} = \mathbf{0}$ must be true for this set of vectors. To begin, the following equation is used:

$$\mathbf{v}^T\mathbf{b} = \mathbf{v}^T \left[ \ddot{\mathbf{x}}_{m_{open}} - \mathbf{\Lambda}_*^{-1}\mathbf{f}_* - \mathbf{\Lambda}_*^{-1} \left( \mathbf{\Lambda}_*^{-1} + \mathbf{\Lambda}_{m-1}^{-1} \right)^{-1} \left( \ddot{\mathbf{x}}_{m-1_{open}} - \mathbf{\Lambda}_*^{-1}\mathbf{f}_* \right) \right]. \quad (4.31)$$

Using Lemma 1 again, this also can be rewritten as follows:

$$\begin{aligned} \mathbf{v}^T\mathbf{b} &= \mathbf{v}^T \left\{ \ddot{\mathbf{x}}_{m_{open}} - \mathbf{\Lambda}_*^{-1}\mathbf{f}_* - \mathbf{\Lambda}_*^{-1} \left[ \mathbf{\Lambda}_* - \mathbf{\Lambda}_*\mathbf{J}_{m-1} \left( \mathbf{H}_{m-1} + \mathbf{J}_{m-1}^T\mathbf{\Lambda}_*\mathbf{J}_{m-1} \right)^{-1} \right. \right. \\ &\qquad \left. \left. \cdot \, \mathbf{J}_{m-1}^T\mathbf{\Lambda}_* \right] \left( \ddot{\mathbf{x}}_{m-1_{open}} - \mathbf{\Lambda}_*^{-1}\mathbf{f}_* \right) \right\}, \quad (4.32) \\ &= \mathbf{v}^T \left[ \ddot{\mathbf{x}}_{m_{open}} - \ddot{\mathbf{x}}_{m-1_{open}} + \mathbf{J}_{m-1} \left( \mathbf{H}_{m-1} + \mathbf{J}_{m-1}^T\mathbf{\Lambda}_*\mathbf{J}_{m-1} \right)^{-1} \right. \\ &\qquad \left. \cdot \, \mathbf{J}_{m-1}^T\mathbf{\Lambda}_* \left( \ddot{\mathbf{x}}_{m-1_{open}} - \mathbf{\Lambda}_*^{-1}\mathbf{f}_* \right) \right]. \quad (4.33) \end{aligned}$$

Since, by definition of the null space, $\mathbf{J}_{m-1}^T\mathbf{v} = \mathbf{0}$, this simplifies even further:

$$\mathbf{v}^T\mathbf{b} = \mathbf{v}^T \left( \ddot{\mathbf{x}}_{m_{open}} - \ddot{\mathbf{x}}_{m-1_{open}} \right). \quad (4.34)$$

Using Eq. (2.5) it is rewritten as follows:

$$\begin{aligned} \mathbf{v}^T\mathbf{b} &= \mathbf{v}^T \left( \ddot{\mathbf{x}}_m + \mathbf{J}_m\mathbf{H}_m^{-1}\mathbf{J}_m^T\mathbf{f}_m - \ddot{\mathbf{x}}_{m-1} - \mathbf{J}_{m-1}\mathbf{H}_{m-1}^{-1}\mathbf{J}_{m-1}^T\mathbf{f}_{m-1} \right), \quad (4.35) \\ &= \mathbf{v}^T \left( \ddot{\mathbf{x}}_m - \ddot{\mathbf{x}}_{m-1} \right). \quad (4.36) \end{aligned}$$

The right hand side is indeed zero because the closed-chain accelerations of all of the chains are equal [and equal to the reference member acceleration as stated for Eq. (2.7), $\ddot{\mathbf{x}}_k = \mathbf{a}_r$]. Therefore the existence of solutions is proven.

The same method for characterizing the singular solutions may still be used on the system of equations when the system contains more than two manipulators in singular configurations. Eq. (4.20) is manipulated, in this case, to obtain a system of equations which will now grow linearly with the number of singular manipulators. The resulting matrix will be singular when the null spaces of any pair of singular Jacobians overlap; that is, the singular "directions" of any two manipulators coincide.

### 4.5.2 Uniqueness of Motion

To show that for the case where two manipulators are in singular configurations, the solution for the reference member acceleration, $\mathbf{a}_r$, is still unique, a previous form of the dynamic equations from Eq. (4.13) is needed:

$$\begin{bmatrix} -\mathbf{\Lambda}_* & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{\Lambda}_{m-1}^{-1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{\Lambda}_m^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{a}_r \\ \mathbf{f}_{m-1} \\ \mathbf{f}_m \end{bmatrix} = \begin{bmatrix} -\mathbf{f}_* \\ \ddot{\mathbf{x}}_{m-1_{open}} \\ \ddot{\mathbf{x}}_{m_{open}} \end{bmatrix}. \tag{4.37}$$

From this set of equations, the following three equations for $\mathbf{a}_r$ can be written:

$$\begin{aligned} \mathbf{a}_r &= \mathbf{\Lambda}_*^{-1}\left(\mathbf{f}_* + \mathbf{f}_{m-1} + \mathbf{f}_m\right), && (4.38) \\ &= -\mathbf{\Lambda}_{m-1}^{-1}\mathbf{f}_{m-1} + \ddot{\mathbf{x}}_{m-1_{open}}, \text{ and} && (4.39) \\ &= -\mathbf{\Lambda}_m^{-1}\mathbf{f}_m + \ddot{\mathbf{x}}_{m_{open}}. && (4.40) \end{aligned}$$

The previous result concerning the existence of solutions indicates that if $\mathbf{f}_m$ satisfies these equations, then $\mathbf{f}'_m = \mathbf{f}_m + \mathbf{v}$ also satisfies these equations where $\mathbf{v}$ is any vector in the null space of $\mathbf{A}$ as defined above. By substituting $\mathbf{f}'_m$ into Eq. (4.40), the following equation is obtained:

$$\begin{aligned} \mathbf{a}'_r &= -\mathbf{\Lambda}_m^{-1}\mathbf{f}'_m + \ddot{\mathbf{x}}_{m_{open}}, && (4.41) \\ &= -\mathbf{\Lambda}_m^{-1}\left(\mathbf{f}_m + \mathbf{v}\right) + \ddot{\mathbf{x}}_{m_{open}}. && (4.42) \end{aligned}$$

Since $\mathbf{\Lambda}_m^{-1}\mathbf{v} = \mathbf{0}$, then

$$\begin{aligned} \mathbf{a}'_r &= -\mathbf{\Lambda}_m^{-1}\mathbf{f}_m + \ddot{\mathbf{x}}_{m_{open}}, && (4.43) \\ &= \mathbf{a}_r. && (4.44) \end{aligned}$$

Therefore, the reference member acceleration has not changed. To find the associated force at the tip of the other singular chain, $\mathbf{f}'_{m-1}$, this result is substituted into Eq. (4.38) which yields the following equalities:

$$\mathbf{a}_r = \mathbf{\Lambda}_*^{-1}\left(\mathbf{f}_* + \mathbf{f}_{m-1} + \mathbf{f}_m\right) = \mathbf{\Lambda}_*^{-1}\left(\mathbf{f}_* + \mathbf{f}'_{m-1} + \mathbf{f}_m + \mathbf{v}\right), \tag{4.45}$$

which implies that

$$\mathbf{f}'_{m-1} = \mathbf{f}_{m-1} - \mathbf{v}. \tag{4.46}$$

This result shows that any components of force added to the tip of one singular manipulator in the "direction" of the singularity, are compensated for by equal and opposite components of force at the tip of the other singular manipulator, and the acceleration of the reference member is still unique.

## 4.6 Summary and Conclusions

A new approach to the solution of the dynamics for the simulation problem has been presented in this chapter for a multiple-manipulator system grasping a common object where an arbitrary number of manipulators are in singular configurations. Disregarding an algorithm to compute open chain dynamics such as the one presented in Chapter 2, the efficient algorithm presented for computing the tip forces and accelerations required in this problem has a computational complexity of $O(s^3)$ where $s$ is the number of the manipulators that are in singular configurations. This is a significant improvement over previous results in which the complexity was $O(m^3)$.

With respect to the algorithms presented in Chapter 2, the simpler set of equations, Eqs. (2.7) and (2.8), were used as the basis for for the algorithm developed in this chapter. It should be emphasized that a version of the resulting algorithms to match the efficient procedure developed in Section 2.3 can be obtained by performing the same procedures presented in this chapter but starting with Eqs. (2.16) and (2.20) in place of Eqs. (2.7) and (2.8).

Several special cases were also examined that apply to systems that are used extensively in research. Since the presence of singularities in a system of manipulators that are grasping a common object is undesirable, the cases with only one or two singularities were also emphasized. In addition, the case for a dual-arm system was also closely examined, and a robust solution procedure was found that required fewer total computations than previous algorithms which did not take singular chains into account. This is a particularly useful result because systems such as these are receiving more research attention recently.

In the context of parallelization of the simulation algorithm that was presented in Chapter 3, the effort to reduce the amount of extra computation performed by these methods becomes even more important. Remember that the RMA computation in the previous algorithm is a computation that must still be performed serially (or redundantly in parallel), and has a significant negative effect on the resulting speedup that can be achieved even under ideal conditions. It is very important to minimize this serial computation even though it may be a small fraction of the computation that is parallelizable. Due to the nature of the computations when chains are singular, tip forces exerted on singular chains must also be computed serially along with the reference member computation. The results in this chapter describe a very efficient method for performing these computations.